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WAVE FRONT ANALYSIS OF A PLANE COMPRESSIONAL PULSE SCATTERED BY A CYLINDRICAL ELASTIC INCLUSION

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Abstract—The plane-strain problem of a stress pulse striking an elastic circular cylindrical inclusion embedded in an infinite elastic medium is treated. The method used determines dominant stress singularities that arise at wave fronts from the focusing of waves refracted into the interior. It is found that a necessary and sufficient condition for the existence of a propagating stress singularity is that the incident pulse has a step discontinuity at its front. The asymptotic wave front behavior of the first few P and SV waves to focus are determined explicitly and it is shown that the contribution from other waves are less important. In the exterior, it is found that in most composite materials the reflected waves have a singularity at their wave front which depends on the angle of reflection. Also the wave front behavior of the first few singular transmitted waves is given explicitly.

The analysis is based on the use of a Watson-type lemma, developed here, and Friedlander's method[5]. The lemma relates the asymptotic behavior of the solution at the wave front to the asymptotic behavior of its Fourier transform on time for large values of the transform parameter. Friedlander's method is used to represent the solution in terms of angularly propagating wave forms. This method employs integral transforms on both time and θ, the circumferential coordinate. The θ inversion integral is asymptotically evaluated for large values of the time transform parameter by use of appropriate asymptotics for Bessel and Hankel functions and the method of stationary phase. The Watson-type lemma is then used to determine the behavior of the solution at singular wave fronts.

The Watson-type lemma is generally applicable to problems which involve singular loadings or focusing in which wave front behavior is important. It yields the behavior of singular wave fronts whether or not the singular wave is the first to arrive. This application extends Friedlander's method to an interior region and physically interprets the resulting representation in terms of ray theory.

NOMENCLATURE

Dimensional parameters:

a	radius of inclusion
c_{d_1}	dilatation wave velocity in inclusion
c_{d_2}	dilatation wave velocity outside inclusion
c_{s_1}	shear wave velocity in inclusion
c_{s_2}	shear wave velocity outside inclusion
$\mathbf{r}$	radial coordinate
$\mathbf{t}$	time
$\mathbf{u}$	radial displacement
$\mathbf{v}$	circumferential displacement
$\mathbf{x}$	cartesian coordinate
$\mathbf{y}$	cartesian coordinate
λ_1, μ_1	Lame' elastic constants of material in inclusion

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λ_2, μ_2 Lame' elastic constants of material outside inclusion
ρ_1 density of material in inclusion
ρ_2 density of material outside inclusion
$\boldsymbol{\sigma}_{r_1}$ radial stress in inclusion
$\boldsymbol{\sigma}_{r_2}$ radial stress outside inclusion
$\boldsymbol{\sigma}_0$ stress amplitude of incident wave
$\boldsymbol{\tau}_{r\theta_1}$ shear stress in inclusion
$\boldsymbol{\tau}_{r\theta_2}$ shear stress outside inclusion.

Dimensionless parameters used in analysis ($\alpha = 1, 2$):

$c = c_{d_1}/c_{d_2}$
$k = c_{d_1}c_{s_2}/c_{d_2}c_{s_1}$
$k_\alpha = c_{d_\alpha}/c_{s_\alpha}$
$r = \mathbf{r}/a$
$t = \mathbf{t}c_{d_2}/a$
$u_\alpha = \mathbf{u}_\alpha/a$
$v_\alpha = \mathbf{v}_\alpha/a$
$\mu = \mu_1/\mu_2$
$\rho = \rho_1/\rho_2$
$\sigma_{r_\alpha} = \boldsymbol{\sigma}_{r_\alpha}/(\lambda_2 + 2\mu_2)$
$\sigma_0 = \boldsymbol{\sigma}_0/(\lambda_2 + 2\mu_2)$
$\tau_{r\theta_\alpha} = \boldsymbol{\tau}_{r\theta_\alpha}/(\lambda_2 + 2\mu_2)$.

1. INTRODUCTION

The purpose of this investigation was to determine the nature of stress singularities that occur when a plane dilatational wave impinges on an elastic circular cylindrical inclusion embedded in an infinite elastic medium (Fig. 1). The specific incident pulse considered has a step function time dependence. There has been interest in this problem in recent years because its solution provides information about the behavior of individual fibers in a composite material subjected to impact loading. Singular stresses which arise from focusing

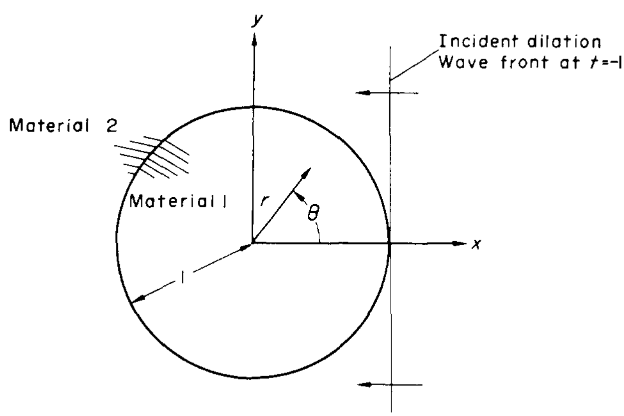

Fig. 1. Plane dilation wave propagating in the exterior region impinges on a circular cylindrical inclusion.

in fiber-reinforced composites can cause separation at the fiber–matrix interface and subsequent loss of load carrying capabilities in the composite.

Papers written on this problem are of two types. Papers by Ko[1] and by Ting and Lee[2] have analyzed this problem to determine the scattering effect of the inclusion on the incident wave. Their eventual goal was to determine the dispersive effect of an array of such inclusions on a stress pulse. While these papers briefly mention that focusing occurs, neither attempts to analyze the nature of the resulting singularities that occur after focusing. Achenbach *et al.*[3], however, are predominantly interested in focusing effects and analyze the first wave front (dilatational) to focus by successive use of wave front analysis, similar to geometric optics, and Poisson's integral representation of the solution to the wave equation.

This investigation corroborates and extends the findings of Achenbach *et al.*, by use of a more general method of treating stress singularities. In addition to those wave fronts treated by Achenbach *et al.*, the wave fronts associated with the following waves were investigated: the shear waves, the diffracted waves, the Stonely interface wave, reflected exterior waves, transmitted (doubly refracted), exterior waves, and the interior refracted wave after many reflections from the interface.

The method of analysis presented here uses a Watson-type lemma which relates the asymptotic behavior of the solution at the wave front to the asymptotic behavior of its Fourier transform on time for large values of the transform parameter. This lemma provides a generalization of Knopoff and Gilbert's technique[4]. Their method is essentially limited to the first wave front to arrive while the method presented here treats subsequent wave fronts as well. This is especially important in problems involving focusing as it is often the later arriving waves which have focused and are singular. Also in the analysis, Friedlander's technique[5] is used to represent the solution in terms of angularly propagating wave forms. Friedlander's technique has been used in exterior regions by Miklowitz[6], Peck and Miklowitz[7], Gilbert[8], and Gilbert and Knopoff[9], to solve problems in which wave pulses are scattered by cylindrical holes or rigid inclusions which cannot directly transmit waves. Chen[10] uses a similar representation to solve a composite problem with an inclusion that transmits waves, but for a periodic excitation of high frequency rather than a transient problem of the type analyzed here. The use of the aforementioned Watson-type lemma translates the high frequency asymptotics of Chen's type into wave front behavior and is a vehicle for interpreting Friedlander's representation for an interior region.

2. METHOD OF ANALYSIS

Statement of problem

The problem is formulated in terms of the displacement potentials ϕ_α and ψ_α, $\alpha = (1, 2)$, where the subscripts 1 and 2 refer to the inner, $r < 1$, and outer, $r > 1$, solids, respectively (see Fig. 1). The radial circumferential displacements are related to ϕ_α and ψ_α by the equations

$$u_\alpha = \frac{\partial \phi_\alpha}{\partial r} + \frac{1}{r}\frac{\partial \psi_\alpha}{\partial \theta} \tag{1.a}$$

$$v_\alpha = \frac{1}{r}\frac{\partial \phi_\alpha}{\partial \theta} - \frac{\partial \psi_\alpha}{\partial r}, \tag{1.b}$$

where all quantities have been nondimensionalized, their meaning specified under *Nomenclature*. Let L_x be the differential wave operator defined as

$$L^x \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{1}{x^2}\frac{\partial^2}{\partial t^2}.$$

In the absence of body forces if the displacement potentials satisfy the wave equations

$$\begin{aligned} &L_c[\phi_1] = L_{c/k_1}[\psi_1] = 0 \qquad \text{for } r < 1, \text{ and} \\ &L_1[\phi_2] = L_{1/k_2}[\psi_2] = 0 \qquad \text{for } r > 1, \end{aligned} \tag{2}$$

where the dimensionless speeds of the dilatation (P type) and shear (SV type) waves are c and c/k_1 in the inclusion, and 1 and k_2^{-1} in the exterior region, then the displacements given by equations (1.a) and (1.b) are a solution to the displacement equations of motion. In addition, the stresses are related to the potentials by the relations

$$\begin{aligned}
\sigma_{r_1} &= c^2\rho\nabla^2\phi_1 + \frac{2c^2\rho}{k_1^2}\left(-\frac{1}{r}\frac{\partial\phi_1}{\partial r} - \frac{1}{r^2}\frac{\partial^2\phi_1}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial\psi_1}{\partial\theta} + \frac{1}{r}\frac{\partial^2\psi_1}{\partial r\,\partial\theta}\right), \\
\tau_{r\theta_1} &= \frac{c^2\rho}{k_1^2}\left(\frac{2}{r}\frac{\partial^2\phi_1}{\partial r\,\partial\theta} - \frac{2}{r^2}\frac{\partial\phi_1}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2} - \frac{\partial^2\psi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_1}{\partial r}\right), \\
\sigma_{r_2} &= \nabla^2\phi_2 + \frac{2}{k_2^2}\left(-\frac{1}{r}\frac{\partial\phi_2}{\partial r} - \frac{1}{r^2}\frac{\partial^2\phi_2}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial\psi_2}{\partial\theta} + \frac{1}{r}\frac{\partial^2\psi_2}{\partial r\,\partial\theta}\right), \\
\tau_{r\theta_2} &= \frac{1}{k_2^2}\left(\frac{2}{r}\frac{\partial^2\phi_2}{\partial r\,\partial\theta} - \frac{2}{r^2}\frac{\partial\phi_2}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi_2}{\partial\theta^2} - \frac{\partial^2\psi_2}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_2}{\partial r}\right).
\end{aligned} \tag{3}$$

In the exterior region, the solution is separated into scattered and incident parts, denoted by the subscripts *sc* and *inc*, respectively. The incident part is specified to be the step stress dilatation wave

$$\phi_{inc} = \frac{\sigma_0}{2}(t + r\cos\theta)^2 H(t + r\cos\theta) \tag{4}$$

where H is the Heaviside step function. Perfect bonding takes place at the interface between the two solids, i.e. the displacement and the radial and shear stresses are continuous at $\mathbf{r} = 1$. In the exterior domain the scattered waves are outgoing as $r \to \infty$, and in the interior the solution is bounded as $r \to 0$. Lastly, the solution is of period 2π in θ, the angular dimension.

Types of waves generated

The waves that are generated when the pulse strikes the inclusion are of three types: diffracted, reflected and refracted. Essentially, the diffracted wave fronts circle the inclusion and propagate around the interface with some characteristic velocity, e.g. the dilatation wave speed in the exterior region. The reflected waves are waves that are generated in the exterior region when the incident dilatation wave strikes the interface. The refracted waves occur when a wave from one media strikes the interface and generates waves in the second media. For example, the dilatation wave refracted into the interior repeatedly reflects from the interface and with each reflection spews refracted P and SV waves into the exterior (see Fig. 2). The geometry and the analytical nature of these waves will be discussed in detail in later sections.

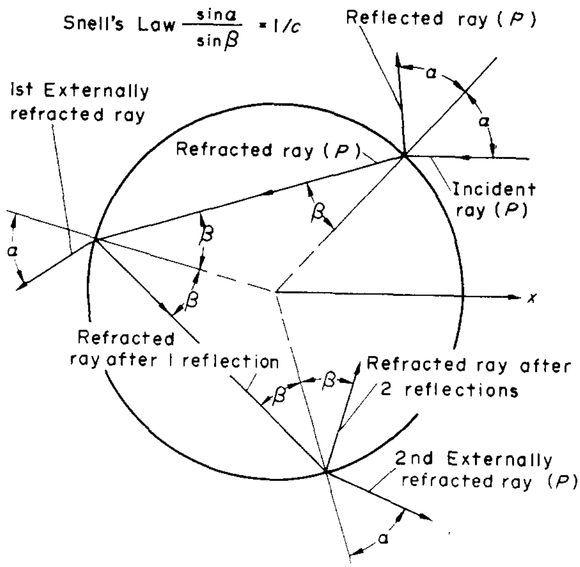

Fig. 2. Ray geometry of the refracted dilatation waves.

Friedlander's representation of the solution

The following discussion parallels that of Miklowitz[6] and Peck and Miklowitz[7] who solved the problem of a stress pulse striking a circular cavity. It differs from their discussion in that Friedlander's representation[5] has been extended to an interior region with the accompanying complications that result from refraction. This representation may be obtained by various means. The most direct method is the application of Poisson's summation formula, which may be stated

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi) e^{2m\pi\xi i} \, d\xi.$$

Applied to the Fourier series representation of a typical response function $f(r, \theta, t)$ this gives

$$\begin{aligned} f(r, \theta, t) &= \sum_{n=-\infty}^{\infty} F(r, n, t) e^{in\theta} \\ &= \sum_{m=-\infty}^{\infty} f^*(r, \theta + 2m\pi, t) \end{aligned} \tag{5}$$

where

$$f^*(r, \theta, t) \equiv \int_{-\infty}^{\infty} F(r, \xi, t) e^{i\xi\theta} \, d\xi. \tag{6}$$

f^* is called the "wave form" of f, and the sum on m in equation (5) is called the "wave sum."

The wave form of the response, f^*, has a physical interpretation. This response corresponds to the disturbance propagating outward in θ. As discussed in the previous section these disturbances are of two types; diffracted and refracted waves. Both propagate along θ. For θ's beyond the wave front, f^* is identically zero. Therefore, for finite t, the sum on m

is finite. Thus, f^* overlaps itself as it propagates in θ and the wave sum is simply the sum of the overlapping responses.

The present problem can be cast in the wave sum form by first finding the Fourier series representation of the solution and then applying the above formulas; however, a more direct method is as follows. Because the only given function in this problem is the incident potential, once its expression in the wave sum form is found, one can simply require that each term of the wave sum for ϕ_{sc}, ψ_α and ϕ_1 satisfy the wave equations

$$\begin{aligned} L_1\phi_{sc}^* = L_{1/k_2}\psi_2^* = 0 \qquad \text{for } r > 1,\ -\infty < \theta < \infty, \\ L_c\phi_2^* = L_{c/k_1}\psi_1^* = 0 \qquad \text{for } r < 1,\ -\infty < \theta < \infty, \end{aligned} \tag{7}$$

and that the boundary conditions are also satisfied term-by-term at $r = 1$ and $-\infty < \theta < \infty$. When the quiescent initial conditions are added together with the appropriate conditions as $r \to \infty$ and $r \to 0$, it is clear that the wave sum of the solution to the above problem is the solution to the original problem. To complete the formulation of the problem in wave sum form, the wave form of the incident potential must be obtained. This is done after the application of integral transforms.

Transformed solution

The Fourier transform on time will be denoted by

$$\bar{f}(r, \theta, \omega) = \int_{-\infty}^{\infty} f(r, \theta, t)\mathrm{e}^{i\omega t}\,\mathrm{d}t \tag{8.a}$$

with the inversion integral

$$f(r, \theta, t) = \frac{1}{2\pi}\int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{f}(r, \theta, \omega)\mathrm{e}^{-i\omega t}\,\mathrm{d}\omega \tag{8.b}$$

where $\gamma > 0$. The subsequent Fourier transform on θ is denoted by

$$\bar{\bar{f}}(r, \nu, \omega) = \int_{-\infty}^{\infty} \bar{f}(r, \theta, \omega)\mathrm{e}^{-i\nu\theta}\,\mathrm{d}\theta \tag{9.a}$$

with the inversion integral

$$\bar{f}(r, \theta, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\bar{f}}(r, \nu, \omega)\mathrm{e}^{i\nu\theta}\,\mathrm{d}\tau. \tag{9.b}$$

The double transform of the wave sum form of the incident potential is found by applying the Poisson summation formula to the Fourier series of $\bar{\phi}_{inc}$. From equation (4), $\bar{\phi}_{inc} = \bar{\phi}_0(\omega)\exp(-i\omega r\cos\theta)$, where $\bar{\phi}_0(\omega) = \sigma_0/(-i\omega)$.[3] The Fourier series may be written as

$$\bar{\phi}_{inc}(r, \theta, \omega) = \sum_{n=-\infty}^{\infty} \bar{\phi}_0(\omega)\mathrm{e}^{-|n|\pi i/2}J_{|n|}(\omega r)\mathrm{e}^{in\theta}$$

where the integral definition of J_n, the Bessel function of the first kind, has been used. Then, applying equation (6) and taking the Fourier transform with respect to θ yields

$$\bar{\bar{\phi}}_{inc}^*(r, \nu, \omega) = \bar{\phi}_0(\omega)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-1/2|\xi|\pi i}J_{|\xi|}(\omega r)\mathrm{e}^{i\theta(\xi-\nu)}\,\mathrm{d}\xi\,\mathrm{d}\theta.$$

Since $J_{|\nu|}(z)$ approaches zero exponentially as $\nu \to \infty$, the Fourier transform theorem gives

$$\bar{\bar{\phi}}_{inc}^*(r, \nu, \omega) = 2\pi\bar{\phi}_0(\omega)e^{-1/2|\nu|\pi i}J_{|\nu|}(\omega r). \tag{10}$$

Applying the double Fourier transform to equations (7) for φ_{sc}^*, ψ_α^* and ϕ_1^* gives the equations for the Bessel functions. Keeping only those solutions that are outgoing as $r \to \infty$ and bounded as $r \to 0$, one obtains

$$\begin{aligned}
\tilde{\tilde{\phi}}_1^*(r, v, \omega) &= A(v, \omega) J_{|v|}(\omega r/c) \\
\tilde{\tilde{\psi}}_1^*(r, v, \omega) &= B(v, \omega) J_{|v|}(\omega r k_1/c) \\
\tilde{\tilde{\phi}}_{sc}^*(r, v, \omega) &= C(v, \omega) H_v^{(1)}(\omega r) \\
\tilde{\tilde{\psi}}_2^*(r, v, \omega) &= D(v, \omega) H_v^{(1)}(\omega r k_2)
\end{aligned} \tag{11}$$

where $H_v^{(1)}(z)$ is the Hankel function of the first kind of order v.

The condition of perfect bonding at the interface is used to determine A, B, C and D. This condition implies that $\mathbf{x} = [A, B, C, D]^T$ satisfies the matrix equation

$$[E]\mathbf{x} = 2\pi\bar{\phi}_0(\omega)\exp\left(\frac{-|v|\pi i}{2}\right)\mathbf{y}$$

where $[E] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4]$

$$\mathbf{e}_1 = \left|\begin{array}{l}
\dfrac{\omega}{c} J'_{|v|}\left(\dfrac{\omega}{c}\right) \\[2ex]
iv J_{|v|}\left(\dfrac{\omega}{c}\right) \\[2ex]
\mu\left[\left(2v^2 - \dfrac{{k_1}^2\omega^2}{c^2}\right) J_{|v|}\left(\dfrac{\omega}{c}\right) - \dfrac{2\omega}{c} J'_{|v|}\left(\dfrac{\omega}{c}\right)\right] \\[2ex]
2iv\mu\left[\dfrac{\omega}{c} J'_{|v|}\left(\dfrac{\omega}{c}\right) - J_{|v|}\left(\dfrac{\omega}{c}\right)\right]
\end{array}\right|$$

$$\mathbf{e}_2 = \left|\begin{array}{l}
iv J_{|v|}\left(\dfrac{\omega k_1}{c}\right) \\[2ex]
-\dfrac{\omega k_1}{c} J'_{|v|}\left(\dfrac{\omega k_1}{c}\right) \\[2ex]
-2iv\mu\left[J_{|v|}\left(\dfrac{\omega k_1}{c}\right) - \dfrac{\omega k_1}{c} J'_{|v|}\left(\dfrac{\omega k_1}{c}\right)\right] \\[2ex]
-\mu\left[\left(2v^2 - \dfrac{{k_1}^2\omega^2}{c^2}\right) J_{|v|}\left(\dfrac{\omega k_1}{c}\right) - \dfrac{2\omega k_1}{c} J'_{|v|}\left(\dfrac{\omega k_1}{c}\right)\right]
\end{array}\right|$$

$$\mathbf{e}_3 = -\left|\begin{array}{l}
\omega H_v^{(1)\prime}(\omega) \\
iv H_v^{(1)}(\omega) \\
(2v^2 - \omega^2 k_2^2) H_v^{(1)}(\omega) - 2\omega H_v^{(1)\prime}(\omega) \\
2iv[\omega H_v^{(1)\prime}(\omega) - H_v^{(1)}(\omega)]
\end{array}\right|$$

$$\mathbf{e}_4 = \left|\begin{array}{l}
-iv H_v^{(1)}(\omega k_2) \\
\omega k_2 H_v^{(1)\prime}(\omega k_2) \\
2iv[H_v^{(1)}(\omega k_2) - \omega k_2 H_v^{(1)\prime}(\omega k_2)] \\
(2v^2 - \omega^2 k_2^2) H_v^{(1)}(\omega k_2) - 2\omega k_2 H_v^{(1)\prime}(\omega k_2)
\end{array}\right|$$

$$\mathbf{y} = \begin{vmatrix} \omega J'_{|n|}(\omega) \\ i\nu J_{|\nu|}(\omega) \\ (2\nu^2 - \omega^2 k_2^2) J_{|\nu|}(\omega) - 2\omega J'_{|\nu|}(\omega) \\ 2i\nu[\omega J'_{|\nu|}(\omega) - J_{|\nu|}(\omega)] \end{vmatrix}$$

which uniquely determines $\mathbf{x}$. From symmetry, u_α^*, $\alpha = 1, 2$, must be even functions of θ and v_α^* must be odd functions of θ which implies $\tilde{\bar{u}}_\alpha^*$ are even functions of ν and $\tilde{\bar{v}}_\alpha^*$ odd, Hence, the absolute value signs in the above equations are dropped and, instead of (9.b). the half range inversion integrals

$$\begin{aligned} \bar{u}_\alpha^*(r, \theta, \omega) &= \frac{1}{\pi}\int_0^\infty \tilde{\bar{u}}_\alpha^*(r, \nu, \omega)\cos(\nu\theta)\,\mathrm{d}\nu \\ \bar{v}_\alpha^*(r, \theta, \omega) &= \frac{i}{\pi}\int_0^\infty \tilde{\bar{v}}_\alpha^*(r, \nu, \omega)\sin(\nu\theta)\,\mathrm{d}\nu, \end{aligned} \tag{12}$$

$\alpha = 1, 2$ are used where $\tilde{\bar{u}}_\alpha^*$ and $\tilde{\bar{v}}_\alpha^*$ are given by

$$\begin{aligned} \tilde{\bar{u}}_\alpha^* &= \frac{\mathrm{d}}{\mathrm{d}r}\tilde{\bar{\phi}}_\alpha^* + \frac{i\nu}{r}\tilde{\bar{\psi}}_\alpha^* \\ \tilde{\bar{v}}_\alpha^* &= \frac{i\nu}{r}\tilde{\bar{\phi}}_\alpha^* - \frac{\mathrm{d}}{\mathrm{d}r}\tilde{\bar{\psi}}_\alpha^*. \end{aligned} \tag{13}$$

A Watson-type lemma for the Fourier transform

Since the governing equations of dynamic elasticity are hyperbolic, any non-stationary discontinuity must occur at wave fronts. In order for singularities from focusing to occur the wave front must be converging. Often the first wave to arrive at a point is not singular while a later wave caused, for example, by a reflection from a concave interface will have converged and hence is singular when it arrives at the point. In this section a method is prescribed for determining the wave front behavior of later arriving singular waves from the integral expressions for the solution without completely evaluating the integrals in closed form.

The solution near the wave fronts is obtained by applying a Watson-type lemma to its Fourier time transform. Essentially, the lemma states that if $\bar{f}(\omega, \mathbf{x})$ is the Fourier transform of $f(t, \mathbf{x})$, $\mathbf{x}$ a position vector, and $f(t, \mathbf{x})$ is the sum of several types of singular functions that commonly occur in focusing, then the Fourier transform of the most singular one will dominate asymptotically as $\omega \to \infty$. Furthermore, this result holds independent of the time of arrival of the various singular waves that compose $f(t, \mathbf{x})$. This is usually not true for Watson's lemma as it is normally stated for the Laplace transform of a function, which is why it is not in general applicable to focusing problems. Preliminary analysis using stationary phase approximations to find the high frequency behavior of the solution helped suggest the types of singularities that are considered in the following lemma. Formally, *Lemma:* Let the Fourier transform of the function $f(t, \mathbf{x})$ be defined as

$$\bar{f}(\omega, \mathbf{x}) = \int_{-\infty}^\infty \mathrm{e}^{i\omega t} f(t, \mathbf{x})\,\mathrm{d}t.$$

Suppose $f(t, \mathbf{x}) = \sum_{n=0}^{4} f_n(t, \mathbf{x})$ where

$$f_0(t, \mathbf{x}) = a_0(\mathbf{x})(t - t_0(\mathbf{x}))^{-b}H(t - t_0(\mathbf{x})),\ b \in (0, 1)$$
$$f_1(t, \mathbf{x}) = a_1(\mathbf{x})(t_1(\mathbf{x}) - t)^{-d}H(t_1(\mathbf{x}) - t),\ d \in (0, 1)$$
$$f_2(t, \mathbf{x}) = \begin{cases} a_2(\mathbf{x}) \ln|t - t_2(\mathbf{x})| & \text{for} \quad |t - t_2(\mathbf{x})| < 1 \\ 0, \quad \text{for} \quad |t - t_2(\mathbf{x})| > 1 \end{cases}$$
$$f_3(t, \mathbf{x}) = a_3(\mathbf{x})H(t - t_3(\mathbf{x}))$$
$$f_4(t, \mathbf{x}) = h(t, \mathbf{x})H(t - t_4(\mathbf{x})),$$

f_4 continuous and

$$\left|\frac{\partial f_4}{\partial t}\right| < M_1 \quad \text{for } t < T \quad \text{and} \quad |h| < M_2 t^n, \quad \text{for } t > T,$$

where M_2 is independent of $\mathbf{x}$ and t, and $n = (0, 1, 2, \ldots)$. Then

$$\bar{f}(\omega, \mathbf{x}) = \frac{a_0(\mathbf{x})\Gamma(1 - b)e^{i\omega t_0(\mathbf{x})}}{(-i\omega)^{1-b}} + \frac{a_1(\mathbf{x})\Gamma(1 - d)e^{i\omega t_1(\mathbf{x})}}{(i\omega)^{1-d}}$$
$$+ \frac{\pi a_2(\mathbf{x})e^{i\omega t_2(\mathbf{x})}}{-\omega} + \frac{a_3(\mathbf{x})e^{i\omega t_3(\mathbf{x})}}{-i\omega} + o(\omega^{-1}) \quad \text{as} \quad \omega \to \infty.$$

To prove this the transforms of f_0, f_1, f_2 and f_3 are directly calculated. To show that $\bar{f}_4(\omega)$ is $o(\omega^{-1})$, the transform of f_4 is integrated once by parts and the Reimann–Lebesque lemma applied to the resulting integral for $Im\{\omega\} > 0$. The result holds by analytic continuation as $Im\{\omega\} \to 0$ and $\omega \to +\infty$.

To use the above lemma it is assumed that the only singularities present are of the type stated. Hence, if it is found, for example, that

$$\bar{f}(\omega, r, \theta) \sim \frac{m(r, \theta)e^{i\omega t_0(r,\theta)}}{\omega} \qquad \omega \to \infty,$$

the above lemma implies

$$f(t, r, \theta) \sim \frac{-m(r, \theta)}{\pi} \ln|t - t_0(r, \theta)|$$

as $t \to t_0$. This lemma is applied in the next chapter to the focusing of refracted waves in the following manner. The Fourier inversion integral on θ of equation (9.b) is used to give an integral representation of the Fourier (time) transform of the solution. This integral is then approximated asymptotically for large values of the parameter ω using the method of stationary phase, and the lemma applied to the resulting expressions.

3. WAVEFRONT ANALYSIS

Introduction

When the stress pulse strikes the inclusion, the discontinuity in material properties at the interface and the shape of the inclusion cause the refracted rays to intersect either on their first pass across the inclusion or on their second pass following a reflection from the interface. Whether the first or the latter event occurs depends on the order of wave speeds of the two materials. While there are six possible wave speed orders only two will be analyzed.

They are:

(i) $c > c/k_1 > 1 > 1/k_2 \Leftrightarrow c > k_1$
(ii) $1 > 1/k_2 > c > c/k_1 \Leftrightarrow c < k_2^{-1}$.

Case (i) is of major interest in composite materials since it is usual for the fiber to be stiffer than the matrix material and hence have faster wave speeds. For this reason it will be analyzed in detail. Other wave speed orders can be analyzed using the methods devised here for these two cases. Case (ii) is especially instructive since the wave speeds are ordered completely differently from (i) and represent the opposite physical case of a soft inclusion and will be briefly analyzed. For a detailed analysis of this case see Ref. [11].

A caustic is an envelope of converging rays. When a ray touches a caustic, focusing or unfocusing can occur. When focusing occurs singular stresses are found at the wave front after it has passed through the caustic. Since the balance of momentum at the wave front yields

$$n_j[\sigma_{ij}] = -\rho s\left[\frac{\partial u_i}{\partial t}\right],$$

where jumps in field quantities are indicated by the usual brackets, and where n_j is the jth component of the unit vector normal to the surface of discontinuity and s is the speed of the wave, the nature of the stress discontinuities may be disclosed by simply calculating the discontinuities in the velocity vector.

The focusing of a refracted dilatation wave for $c < 1$

Consider the directly refracted interior P wave when the speed c is less than the dimensionless exterior P wave speed, 1. If α is the angle of incidence of the exterior P wave which strikes the interface, the angle of refraction of the generated P wave is β, where α and β are related by Snell's law, i.e.

$$c \sin \alpha = \sin \beta. \tag{14}$$

Using simple geometry, the envelope of converging refracted P rays is depicted in Fig. 3 where, in addition to this caustic, three of its generating incident and refracted ray pairs are shown.

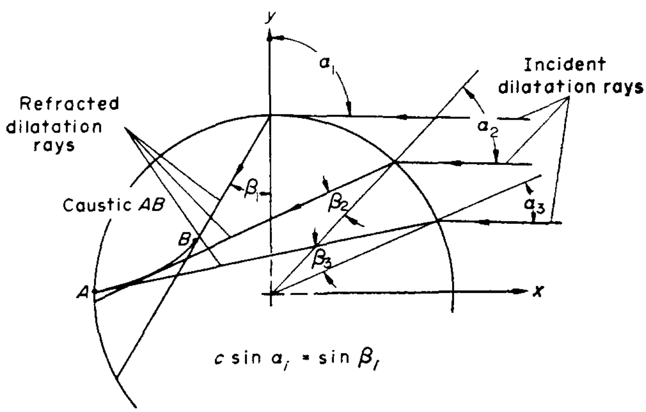

Fig. 3. Refracted dilatation rays and caustic for $c = \frac{1}{2}$, $2\pi > \theta \geq 0$.

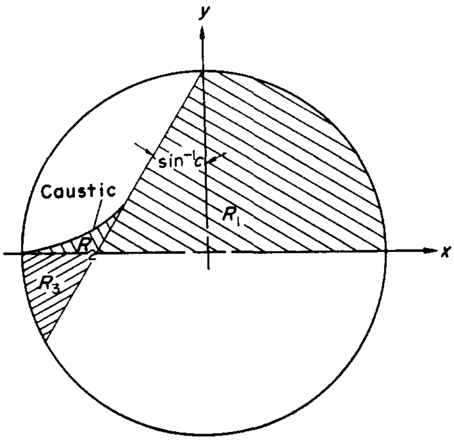

Fig. 4. Regions R_1, R_2 and R_3 for $c = \frac{1}{2}$, $\theta > 0$.

Consider how the solution varies along a refracted ray that touches the caustic. Let d_{1p} be the distance traveled along the ray from the interface where it was generated to a point (r, θ) and h_{1p} be the distance along the ray to this caustic. From Fig. 3, it is clear that there are three regions in which (r, θ) might lie:

(1) $d_{1p} < h_{1p}$ and there is only one ray per point,

(2) $d_{1p} \sim h_{1p}$, a region that includes the caustic as well as a transition zone where there are two rays per point,

(3) $d_{1p} > h_{1p}$ and only one ray per point.

Call the collections of all such points for all the directly refracted dilatation rays R_1, R_2 and R_3, respectively. These regions are portrayed for $\theta > 0$ and $c = \frac{1}{2}$ in Fig. 4. When a ray touches this caustic it picks up a singularity which propagates along its wave front. Thus points in R_2 and R_3 see singular stresses from these refracted rays. Consider the following mathematical analysis that verifies the above discussion and reveals the orders of these singularities.

Let the part of the radial velocity, $\dot{u}_1^*$, that propagates with the dilatation speed c be denoted as $\dot{u}_{1(dil)}^*$. From the elementary properties of the wave equation, contributions from ϕ_1^* propagate with this velocity while contributions from ψ_1^* propagate with the shear velocity c/k_1, thus equations (13) and (11) imply

$$\tilde{\bar{u}}_{1(dil)}^*(r, \nu, \omega) = -\frac{i\omega^2}{c} A(\nu, \omega) J_\nu' \left(\frac{\omega r}{c}\right). \tag{15}$$

Equation (12) implies

$$\bar{u}_{1(dil)}^*(r, \theta, \omega) = \frac{1}{\pi} \int_0^\infty \tilde{\bar{u}}_{1(dil)}^*(r, \nu, \omega) \cos \nu\theta \, d\nu.$$

The asymptotic behavior of this quantity for large ω is of interest. Once it is known, the Watson-type lemma for the Fourier time transform will be used. For $\omega = \omega_1 + i\omega_2$, $\omega_2 > 0$ and ω_1 large, the change of variable $\nu = s\omega$ yields

$$\bar{u}_{1(dil)}^*(r, \theta, \omega) = \frac{1}{2\pi} \int_{C_s} \omega \tilde{\bar{u}}_{1(dil)}^*(r, s\omega, \omega)[e^{i\omega s\theta} + e^{-is\omega\theta}] \, ds, \tag{16}$$

where C_s is a line from the origin to ∞ inclined at an angle of $-\tan^{-1}\omega_2/\omega_j$. Consider the contribution from the term $e^{i\omega s\theta}$ in (16), which will be called $\bar{u}^{*+}_{1(dil)}$, i.e.

$$\bar{u}^{*+}_{1(dil)}(r, \theta, \omega) = \frac{1}{2\pi}\int_{C_s} \omega\tilde{\bar{u}}^{*}_{1(dil)}(r, s\omega, \omega)e^{is\omega\theta}\, ds. \tag{17}$$

The reason for neglecting the contribution of $e^{-is\omega\theta}$ in equation (16) will be made clear later.

As $\omega_1 \to \infty$, C_s is equivalent to a path along the real s-axis which is indented below any poles of the integrand that might lie on the real s-axis. With this understanding, equation (17) is written

$$\bar{u}^{*+}_{1(dil)}(r, \theta, \omega) = \frac{1}{2\pi}\int_0^{\infty} \omega\tilde{\bar{u}}^{*}_{1(dil)}(r, s\omega, \omega)e^{is\omega\theta}\, ds. \tag{18}$$

In the following discussion this integral is evaluated asymptotically for large ω in a manner similar to that used by Chen[10]. For ω large, the asymptotics given in Ref. [11] imply that for $s\varepsilon[0, \min(1, 1/c))$, i.e. $0 \le s < \min(1, 1/c) \equiv$ smaller of either 1 or $1/c$, $A(s\omega, \omega)$ may be expanded in a geometric series, each term of which corresponds to a different refracted ray. It is found, for $s\varepsilon[0, \min(1, 1/c))$, that

$$A(s\omega, \omega) \sim A_0(s, \omega)[1 - \delta_{11}(s)M_{1/c}(s) + \Gamma_1] \tag{19}$$

where

$$A_0(s, \omega) = \frac{2\pi\sigma_0(m_{1/c}(s)m_1(s))^{1/2}a_0(s)\exp[i\omega(-s\pi/2 + \psi_{1/c}(s) - \psi_1(s))]}{-i\omega^3\,\delta_{10}(s)}$$

$$a_0(s) = 4e_2 s^2(m_{k_1/c} + m_{k_2}) - 8s^4(m_{k_2} + \mu m_{k_1/c}) - 2e_2(m_{k_1/c}e_2 - m_{k_2}\mu e_1) + 4\mu s^2(m_{k_1/c}e_2 - m_{k_2}e_1),$$

$$\delta_{10}(s) = (m_{1/c}m_{k_1/c} + s^2)(e_2^2 + 4s^2 m_1 m_{k_2}) + \mu[-2s^2(2m_{1/c}m_{k_1/c} + e_1)(2m_1 m_{k_2} + e_2) + \left(\frac{k_1k_2}{c}\right)^2(m_{1/c}m_{k_2} + m_1 m_{k_1/c})] + \mu^2(e_1^2 + 4s^2 m_{1/c}m_{k_1/c})(m_1 m_{k_2} + s^2),$$

$$\delta_{11}\,\delta_{10} = (-m_{1/c}m_{k_1/c} + s^2)(e_2^2 + 4s^2 m_1 m_{k_2}) + \mu[2s^2(2m_{1/c}m_{k_1/c} + e_1)(2m_1 m_{k_2} + e_2) + \left(\frac{k_1k_2}{c}\right)^2(m_1 m_{k_1/c} - m_{1/c}m_{k_2})] + \mu^2(e_1^2 - 4s^2 m_{1/c}m_{k_1/c})(m_1 m_{k_2} + s^2),$$

$$\psi_x(s) = m_x(s) - s\cos^{-1}(s/x),$$

$$M_x(s) = -i\exp(2i\omega\psi_x(s)),$$

$$m_x(s) = \sqrt{|x^2 - s^2|}, \qquad e_1 = \left(2s^2 - \frac{k_1^2}{c^2}\right), \qquad e_2 = (2s^2 - k_2^2).$$

It will be shown that in equation (19), A_0 corresponds to the directly refracted dilatation wave before it reflects for the first time from the interface. $-A_0\,\delta_{11}M_{1/c}$ corresponds to the same wave after one reflection but before the second reflection, and Γ_1 corresponds to other refracted waves. In anticipation of this physical meaning, let $\dot{u}^{*}_{1p}$ be the contribution to the velocity from A_0 where p refers to dilatation. Using the asymptotic forms for the

Bessel functions given in Ref. [11], equations (15, 18 and 19) imply

$$\bar{u}^*_{1p}(r, \theta, \omega) \sim \int_0^{\min(r/c,1)} \frac{H_{1p}(s, r)e^{i\pi/4}}{\sqrt{\omega}} [e^{i\omega f_{d_1}} - ie^{i\omega f_{d_2}}]\, ds \tag{20}$$

where

$$H_{1p}(s; r) = \frac{\sigma_0\, a_0(s)}{r\, \delta_{10}(s)} \left(\frac{m_{1/c}\, m_1 m_{r/c}}{2\pi}\right)^{1/2},$$

$$f_{d_1}(s; r, \theta) = -\frac{s\pi}{2} + s\theta + \psi_{1/c}(s) - \psi_1(s) + \psi_{r/c}(s),$$

$$f_{d_2}(s; r, \theta) = -\frac{s\pi}{2} + s\theta + \psi_{1/c}(s) - \psi_1(s) + \psi_{r/c}(s).$$

Consider the contribution from the second term in the above integrand. A point of stationary phase, s_0, exists provided $f'_{d_2}(s_0;\ r,\ \theta) = 0$, i.e. since $\psi'_x(s) = \sin^{-1}(s/x) - \pi/2$, where $0 \leq \sin^{-1}(s/x) \leq \pi/2$ for $0 \leq s/x \leq 1$, then

$$-\gamma + \beta - \alpha + \theta = 0 \tag{21}$$

where

$$\alpha = \sin^{-1} s_0, \qquad \beta = \sin^{-1} s_0\, c, \qquad \gamma = \sin^{-1} \frac{s_0\, c}{r}. \tag{22}$$

Equation (21) is satisfied if $\theta \geq 0$ and when s_0 is such that α, β and γ have the geometric relationship shown in Fig. 5. Note that α and β satisfy Snell's law, equation (14). Furthermore,

$$f_{d_2}(s_0; r, \theta) = \frac{d_{1p}}{c} + d_1 - 1 = t_{1p}(r, \theta), \tag{23}$$

where d_1 and d_{1p} have the physical meanings illustrated in Fig. 5 and hence t_{1p} is the time of arrival of the refracted P wave at the point (r, θ), thus justifying the nomenclature that has

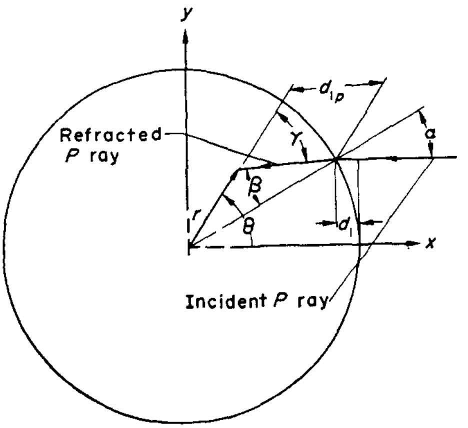

Fig. 5. Refracted dilatation ray for $c < 1$, $d_{1p} < \cos\beta$.

been used. The mathematical restriction that $0 \leq s_0 < \min(r/c, 1)$ physically restricts the incident ray to the positive illuminated zone $0 \leq \alpha < \pi/2$, and restricts the refracted ray so that $d_{1p} < \cos\beta$. Since

$$f''_{d_2}(s_0; r, \theta) = -\frac{c}{r\cos\gamma} + \frac{c}{\cos\beta} - \frac{1}{\cos\alpha} < 0, \qquad \text{for } c < 1, \tag{24}$$

the method of stationary phase implies the contribution from this term is

$$\frac{-iH_{1p}(s_0; r)}{\omega}\sqrt{\frac{2\pi}{-f''_{d_2}(s_0; r, \theta)}}\, e^{i\omega t_{1p}(r,\theta)} \tag{25}$$

for $d_{1p} < \cos\beta$ and $\theta > 0$.

The first term in the integrand of equation (17) has a point of stationary phase, s_0, provided $f'_{d_1}(s_0; r, \theta) = 0$, i.e.

$$\gamma' - \pi + \beta - \alpha + \theta = 0 \tag{26}$$

where α and β are as in equation (22), $\theta \geq 0$ and

$$\gamma' = \sin^{-1}(s_0 c/r). \tag{27}$$

The restriction that $s_0 < r/c$ implies $d_{1p} > \cos\beta$. The angles α and β have the same physical interpretation as before and γ' is the supplement of γ. Again $f_{d_1}(s_0; r, \theta) = t_{1p}$. However,

$$f''_{d_1}(s_0; r, \theta) = (h_{1p} - d_{1p})/\lambda_{1p}, \tag{28}$$

where

$$h_{1p} = \cos^2\beta/(\cos\beta - c\cos\alpha) > \cos\beta,$$
$$\lambda_{1p} = r\cos\alpha\cos\gamma\cos\beta/(\cos\beta - c\cos\alpha) > 0, \qquad \text{for } c < 1,$$

and can be positive or negative depending upon whether d_{1p} is less than or greater than the distance to the caustic, h_{1p}. If the point (r, θ) is restricted to lie on a particular refracted ray specified by the angle of incidence $\alpha_0 = \sin^{-1} s_0$, $s_0 \varepsilon (0, 1)$ and Snell's Law, then on this ray, as d_{1p} increases, five cases can be considered and discussed in terms of the physical description given at the beginning of this section. The cases are:

(i) $(r, \theta)\varepsilon R_1$, $d_{1p} < h_{1p}$. s_0 is the only zero of $f'_{d_1}(s; r, \theta)$ on $(0, \min(r/c, 1))$.
(ii) $(r, \theta)\varepsilon R_2$, $d_{1p} < h_{1p}$ and two zeros exist s_1 and s_0, $s_1 > s_0$, where $f''_{d_1}(s_1; r, \theta) < 0$.
(iii) $d_{1p} = h_{1p}$, $f''_{d_1}(s_0; r, \theta) = 0$, $f'''_{d_1}(s_0, r, \theta) < 0$.
(iv) $(r, \theta)\varepsilon R_2$, $d_{1p} > h_{1p}$ and two zeros exist s_0 and s_2, $s_0 > s_2$, $f''_{d_1}(s; r, \theta) > 0$.
(v) $(r, \theta)\varepsilon R_3$, $d_{1p} > h_{1p}$; s_0 only zero.

For each of these cases $f''_{d_1}(s_0; r, \theta)$ is given by equation (28). It is assumed that s_0, s_1 and s_2 are sufficiently distinct that the method of stationary phase can be separately applied to the three points. Hence, the solutions found for cases (ii) and (iii) do not apply as $d_{1p} \to h_{1p}$, which is why case (iii) must be considered separately. In all the cases except (iii), the points of stationary phase are of order 1. The method of stationary phase is applied and the following asymptotics are found for $\bar{u}^*_{1p}(r, \theta, \omega)$:

$$\frac{H_{1p}(s_0;r)}{-i\omega}\sqrt{\frac{2\pi}{|f''_{d_1}(s_0;r,\theta)|}}\,\mathrm{e}^{i\omega t_{1p}}, \qquad \text{for (i)},$$

$$\frac{H_{1p}(s_0,r)}{-i\omega}\sqrt{\frac{2\pi}{|f''_{d_1}(s_0;r,\theta)|}}\,\mathrm{e}^{i\omega t_{1p}}+\frac{H_{1p}(s_1;r)}{\omega}\sqrt{\frac{2\pi}{|f''_{d_1}(s_1;r,\theta)|}}\,\mathrm{e}^{i\omega f_{d_1}(s_1;r,\theta)}, \qquad \text{for (ii)}$$

$$\frac{H_{1p}(s_0;r)}{\omega}\sqrt{\frac{2\pi}{|f''_{d_1}(s_0;r,\theta)|}}\,\mathrm{e}^{i\omega t_{1p}}+\frac{H_{1p}(s_2;r)}{-i\omega}\sqrt{\frac{2\pi}{|f''_{d_1}(s_2;r,\theta)|}}\,\mathrm{e}^{i\omega f_{d_1}(s_2;r,\theta)}, \qquad \text{for (iv)}. \tag{29}$$

and

$$\frac{H_{1p}(s_0;r)}{\omega}\sqrt{\frac{2\pi}{|f''_{d_1}(s_0;r,\theta)|}}\,\mathrm{e}^{i\omega t_{1p}} \qquad \text{for (v)},$$

as $\omega\to\infty$, $d_{1p}>\cos\beta$, $\theta>0$.

Case (iii) has a higher order stationary phase point. Its contribution is asymptotically

$$\bar{\dot{u}}^*_{1p}(h_{1p},\omega)\sim\frac{H_{1p}(s_0;r)\mathrm{e}^{i\omega t_{1p}+i\pi/4}}{\omega^{5/6}\sqrt[3]{\dfrac{|f'''_{d_1}(s_0;h_{1p})|}{6}}}\,\frac{2\pi}{3\Gamma(2/3)} \qquad \text{as } \omega\to+\infty. \tag{30}$$

Equations (25, 29 and 30) together with the Watson-type lemma imply for $\theta>0$

$$\bar{\dot{u}}^*_{1p}(r,\theta,t)\sim -H_{1p}(s_0;r)\sqrt{\frac{2\pi}{|f''_{d_2}(s_0;r,\theta)|}}\,H(t-t_{1p}) \qquad \text{as } t\to t_{1p}, \tag{31}$$

$d_{1p}<\cos\beta$. And for $d_{1p}>\cos\beta$,

$$\dot{u}^*_{1p}\sim\begin{cases} H_{1p}(s_0;r)\left(\dfrac{2\pi\lambda_{1p}}{h_{1p}-d_{1p}}\right)^{1/2}H(t-t_{1p}),\ d_{1p}\varepsilon R_1,\\ H_{1p}(s_0;r)\left(\dfrac{2\pi\lambda_{1p}}{h_{1p}-d_{1p}}\right)^{1/2}H(t-t_{1p})-H_{1p}(s_1;r)\left(\dfrac{-2}{\pi f''_{d_1}(s_1;r,\theta)}\right)^{1/2}\ln|t-f_{d_1}(s_1;r,\theta)|,\\ \qquad \text{for } d_{1p}\varepsilon R_2,\ d_{1p}<h_{1p},\\ \dfrac{2\pi H_{1p}(s_0;r)}{3\Gamma(2/3)\Gamma(5/6)}\left(\dfrac{6}{|f'''_{d_1}(s_0;h_{1p})|}\right)^{1/3}[(t_{1p}-t)^{-1/6}H(t_{1p}-t)+\sqrt{3}(t-t_{1p})^{-1/6}H(t-t_{1p})],\\ \qquad \text{for } d_{1p}=h_{1p},\\ H_{1p}(s_2;r)\left(\dfrac{2\pi}{f''_{d_1}(s_2;r,\theta)}\right)^{1/2}H(t-f_{d_1}(s_2;r,\theta))-H_{1p}(s_0;r)\left(\dfrac{2\lambda_{1p}}{\pi(d_{1p}-h_{1p})}\right)^{1/2}\ln|t-t_{1p}|\\ \qquad \text{for } d_{1p}\varepsilon R_2,\ d_{1p}>h_{1p},\\ -H_{1p}(s_0;r)\left(\dfrac{2\lambda_{1p}}{\pi(d_{1p}-h_{1p})}\right)^{1/2}\ln|t-t_{1p}|,\ d_{1p}\varepsilon R_3,\\ \qquad \text{where } \left.\dfrac{\mathrm{d}f_{d_1}}{\mathrm{d}s}\right|_{s=s_0,\,s_1,\,s_2}=0, \qquad s_1>s_0>s_2. \end{cases} \tag{32}$$

The physics of this solution is clear. For $d_{1p} < \cos\beta$ the solution along the ray given by α (angle of incidence) $= \sin^{-1} s_0$ is a nonsingular step function. As d_{1p} increases the ray reaches the region R_2 (see Figs. 4 and 5) in which there are two rays per point. The second ray, specified by $\alpha' = \sin^{-1} s_1$ has already touched the caustic and propagates a logarithmic singularity at its wave front which arrives at the point (r, θ) at time $t = f_{d_1}(s_1; r, \theta)$. d_{1p} continues to increase until it is at the caustic, $d_{1p} = h_{1p}$, where a singularity of order $(t - t_{1p})^{-1/6}$ is calculated. After touching the caustic, the ray is again in a region where there are two rays per point; however, this time it is the ray $\alpha = \sin^{-1} s_0$ that has touched the caustic and propagates a logarithmic singularity. Lastly, the ray continues into R_3, another region in which there is only one ray per point and since it has touched the caustic it has a logarithmic singularity at its wave front. This logarithmic singularity at the wave front agrees with that found in Ref. [3] where methods (Poisson's integral representation of the solution to the wave equation) analogous to those used in Friedlander's book[5] were used to investigate the singular nature of the solution. However, in that paper it was incorrectly concluded, as one might from case (ii) by taking an improper limit, that the discontinuity increases beyond bounds like $(h_{1p} - d_{1p})^{-1/2}$ as $d_{1p} \to h_{1p}$. Actually, Friedlander's book[5] discusses the nature of the singularity in the vicinity of the caustic for acoustic waves with a result analogous to that found here.

The right hand side of (32) is written symbolically as

$$\Omega\left(H_{1p}, s_0, \frac{d_{1p} - h_{1p}}{\lambda_{1p}}, t_{1p}\right)$$

where H_{1p} is a refraction coefficient,
s_0 specifies the refracted ray since
α(angle of incidence of external ray) $= \sin^{-1} s_0$,
β(angle of refraction of internal ray) $= \sin^{-1} s_0 c$,
d_{1p} is the distance along the refracted ray from where it was generated at the interface to the point (r, θ),
h_{1p} is the distance along the refracted ray to the caustic, and
t_{1p} is the time of arrival of the wave front traveling along that ray.

Thus for $\theta > 0$,

$$\dot{u}^*_{1p}(r, \theta, t) \sim \begin{cases} -H_{1p}(s_0, r)\sqrt{\dfrac{2\pi}{|f''_{d_2}(s_0; r, \theta)|}}\, H(t - t_{1p}), & \text{for } d_{1p} < \cos\beta, \\ \Omega\left(H_{1p}, s_0, \dfrac{d_{1p} - h_{1p}}{\lambda_{1p}}, t_{1p}\right), & \text{for } d_{1p} > \cos\beta. \end{cases} \tag{33}$$

Recall that (33) is the contribution from the term $e^{i\omega s\theta}$ in (16). The role of the neglected term $e^{-i\omega s\theta}$ in (16) is now clear. Since points of stationary phase existed only for $\theta \geq 0$ for $e^{i\omega s\theta}$, $e^{-i\omega s\theta}$ will have stationary phase points only if $\theta \leq 0$. Thus the solution for negative θ comes from this second term $e^{-i\omega s\theta}$, hence

$$\dot{u}^*_{1(dil)}(r, \theta, t) = \dot{u}^*_{1(dil)}(r, -\theta, t). \tag{34}$$

When $\theta = 0$, both terms $e^{i\omega s\theta}$ and $e^{-i\omega s\theta}$ contribute to the solution. However, since s_0, the point of stationary phase, is zero, an end point of the interval $[0, r/c)$, each term contributes only half as much. Thus the result is the same as in (33) for $\theta = 0$.

Analysis for $c > k_1$

The following notation is adopted with regards to refracted and reflected waves. The subscript 1 or 2 implies that the solution applies to the inner or the exterior region, respectively. Subsequent subscripts of p and/or s yield the "ray history" of the wave and its type, P or SV (shear vertical wave) in the following manner. When the incident exterior P ray strikes the interface it generates refracted interior P and SV waves which will have the subscripts $1p$ and $1s$, and the reflected exterior P and SV rays with subscripts $2p$ and $2s$. When the $1p$ ray strikes the interface after traversing the inclusion it generates rays in the interior denoted by the subscripts $1pp$ (reflected P wave) and $1ps$ (reflected SV wave) and in the exterior denoted by $2pp$ (refracted P wave) and $2ps$ (refracted SV wave). All reflected and refracted rays may be denoted in this manner. In addition, the above subscripts will be used to denote quantities that are associated with a particular ray. d, h and t when subscripted have the following meanings:

- d is the distance along the ray of interest from the interface where it was generated to the point (r, θ).
- h when positive, is the distance along the ray to the caustic, i.e. (r, θ) is on caustic when $d = h$.
- t when subscripted, is the time of arrival of the wavefront propagating along the ray.

Examples of these are d_{1p}, h_{1p} and t_{1p} of the previous section.

For $c > k_1$, the directly refracted P rays diverge on their first pass across the inclusion. However, when they strike the interface for the first time the reflected P and SV rays that are generated converge and form caustics. The following discussion of the generated P rays and their focusing parallels that given earlier for the case $c < 1$.

Using the fact that when a P ray reflects from an interface the angle of reflection equals the angle of incidence and Snell's law, the envelope of converging reflected P rays is depicted in Fig. 6 where, in addition to this caustic, several generating refracted/reflected P ray pairs are shown.

Consider how the solution varies along a reflected ray. d_{1pp} is the distance traveled along the reflected P ray from the interface where it was generated to the point (r, θ) and h_{1pp}

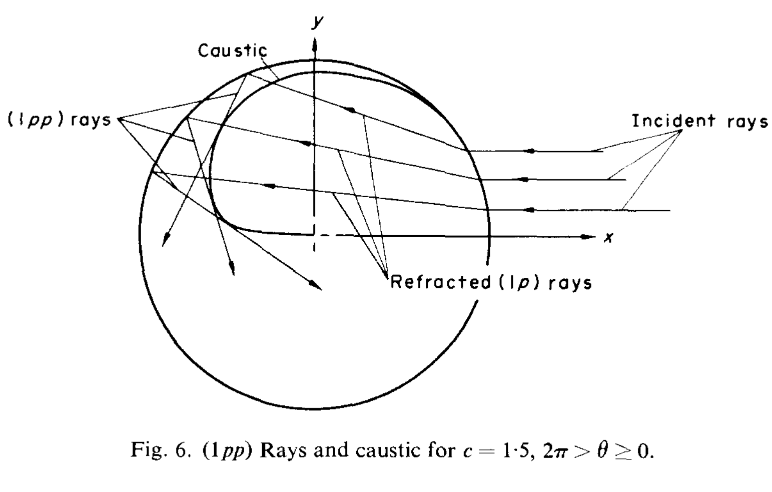

Fig. 6. $(1pp)$ Rays and caustic for $c = 1{\cdot}5$, $2\pi > \theta \geq 0$.

is the distance along this ray to the caustic. From Fig. 6 it is clear that for $\theta > 0$ there are two regions where (r, θ) might lie:

(1) d_{1pp} such that $\theta \leq \pi$. In this region there are two rays per point, one that has touched the caustic and one that has not. Call the collection of all such points for all once reflected P rays R_2^*.

(2) d_{1pp} sufficiently large so that $\theta > \pi$, then there is only one ray per point and it has touched the caustic. Call the collection of all such points R_3^*.

R_2^* and R_3^* are portrayed for $\theta > 0$ and $c = 1{\cdot}5$ in Fig. 7. R_2^* and R_3^* are analogous to R_2 and R_3 discussed earlier and shown in Fig. 4. Points contained in R_2^* and R_3^* experience logarithmic singularities in stresses associated with the arrival of wave fronts that propagate along rays that have touched the caustic. Hence, from Fig. 6, it is clear that the interface, $r = 1$, experiences logarithmic singularities for $\sin^{-1}(1/c) < \theta < 2\pi$ from positively propagating reflected P waves and for $-2\pi < \theta < -\sin^{-1}(1/c)$ from negatively propagating waves. Thus, every point of the physical interface experiences a logarithmic singularity from waves that have reflected once.

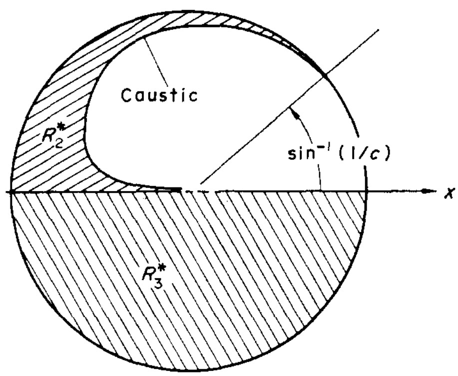

Fig. 7. R_2^* and R_3^* for $2\pi > \theta \geq 0$ and $c = 1{\cdot}5$.

$\dot{u}_{1pp}^*$ and $\dot{v}_{1pp}^*$ are the contributions to the radial and angular velocity from this once reflected P ray. It is found that for $d_{1pp} < \cos\beta$

$$\begin{bmatrix} \dot{u}_{1pp}^*(r, \theta, t) \\ \dot{v}_{1pp}^*(r, \theta, t) \end{bmatrix} \sim \begin{bmatrix} \Omega(H_{1p} \cdot \delta_{11}, s_0, (d_{1pp} - h_{1pp})/\lambda_{1pp}, t_{1pp}) \\ \Omega(-\tan\gamma \cdot H_{1p} \cdot \delta_{11}, s_0, (d_{1pp} - h_{1pp})/\lambda_{1pp}, t_{1pp}) \end{bmatrix} \tag{35}$$

as $t \to t_{1pp}$. H_{1p} and δ_{11} are given in (20) and (19). s_0 is such that

$$\theta - \gamma + 3\beta - \alpha - \pi = 0, \tag{36}$$

where α, β and γ are given in (22). The geometric significance of (36) is analogous to that of (21). d_{1pp} equals $\cos\beta - r\cos\gamma$ and h_{1pp} is equal to $\cos\beta(2c\cos\alpha - \cos\beta)/(3c\cos\alpha - \cos\beta)$ and is always less than $\cos\beta$, and, hence, focusing occurs for all rays. In addition,

$$\lambda_{1pp} = r\cos\gamma\cos\alpha\cos\beta/(3c\cos\alpha - \cos\beta)$$

and

$$t_{1pp} = \frac{d_{1pp}}{c} + \frac{2\cos\beta}{c} + d_1 - 1.$$

For $d_{1pp} > \cos\beta$, all of the reflected P rays have focused and have a logarithmic singularity at the wave front, hence for $d_{1pp} > \cos\beta$ it is found that

$$\begin{bmatrix} \dot{u}^*_{1pp} \\ \dot{v}^*_{1pp} \end{bmatrix} \sim \begin{bmatrix} 1 \\ \tan\gamma' \end{bmatrix} H_{1p}(s_0; r)\,\delta_{11}(s_0)\sqrt{\frac{2\lambda_{1pp}}{\pi(d_{1pp} - h_{1pp})}}\ln|t - t_{1pp}| \text{ as } t \to t_{1pp} \tag{37}$$

$\gamma' = \sin^{-1}(s_0 c/r)$ is the supplement of γ. By comparing these expressions with those for $\dot{u}^*_{1p}$, it is clear that $\delta_{11}(s_0)$ acts as a reflection coefficient.

The first singular shear waves to reach an interior point are the reflected SV waves $\dot{u}^*_{1ps}$ and $\dot{v}^*_{1ps}$ which are generated when the refracted dilatation wave strikes the boundary for the first time.

Analogous to equation (35) it is found that

$$\begin{bmatrix} \dot{u}^*_{1ps} \\ \dot{v}^*_{1ps} \end{bmatrix} \sim \begin{bmatrix} \Omega\left(H_{1s}\cdot(b_1 - \delta_{11}), s_0, \dfrac{d_{1ps} - h_{1ps}}{\lambda_{1ps}}, t_{1ps}\right) \\ \Omega\left(H_{1s}\cdot(b_1 - \delta_{11})\cdot\cot\chi, s_0, \dfrac{d_{1ps} - h_{1ps}}{\lambda_{1ps}}, t_{1ps}\right) \end{bmatrix} \tag{38}$$

as $t \to t_{1ps}$, for $d_{1ps} < \cos\zeta$ where s_0 is such that

$$-\chi + \zeta - \alpha + \theta + 2\beta - \pi = 0,$$

$$\alpha = \sin^{-1} s_0,\ \beta = \sin^{-1}(s_0/c),\ \chi = \sin^{-1}(s_0 c/k_1 r),$$
$$\zeta = \sin^{-1}(s_0 c/k_1).$$

Also,

$$H_{1s} = -\frac{\sigma_0 s b_0}{r\,\delta_{10}}\left(\frac{m_1 m_{k_1/c}}{2\pi m_{k_1 r/c}}\right)^{1/2},$$

$$b_1(s) = \{4se_2(m_{1/c}m_{k_{1/c}} + s^2) - 4s^3(2m_{1/c}m_{k_2} + \mu e_1) - 2se_2(e_2 + 2m_{1/c}m_{k_2}) + 2\mu s(e_1 e_2 + 4s^2 m_{1/c}m_{k_2})\}/b_0$$

where

$$b_0(s) = -\frac{\sigma_0 s b_0(s)}{r\,\delta_{10}(s)}\left(\frac{m_1 m_{k_1/c}}{2\pi m_{k_1 r/c}}\right)^{1/2}.$$

In addition,

$$d_{1ps} = \cos\zeta - r\cos\chi,$$
$$h_{1ps} = (2c\cos\alpha - \cos\beta)\cos^2\zeta/[(c/k_1)\cos\alpha\cos\beta + \cos\zeta] < \cos\zeta,$$
$$\lambda_{1ps} = r\cos\chi\cos\alpha\cos\beta\cos\zeta/[(c/k_1)\cos\alpha\cos\beta + \cos\zeta] > 0,$$
$$t_{1ps} = \frac{k_1 d_{1ps}}{c} + \frac{2\cos\beta}{c} + d_1 - 1.$$

Lastly, e_α, m_x and δ_{11} are given in (19).

In the exterior region two types of waves can be singular when $c > k_1$: reflected waves, and waves refracted into the exterior by internally refracted waves.

When α, the angle of incidence of the exterior P ray, equals $\sin^{-1}(1/c)$, Snell's law, (14), implies that β, the angle of refraction of the interior P ray, is 90° and hence this ray is critically refracted. This situation, in a simpler physical context, is discussed briefly for acoustic waves in Friedlander's book[5]. As is noted there, the reflected waves experience a logarithmic singularity at their wave fronts when $\alpha > \sin^{-1}(1/c)$. Mathematically, the treatment of these waves is analogous to those of previous sections with the difference that s_0, the point of stationary phase, is greater than $1/c$. As in all the previous cases, $s_0 = \sin^{-1}\alpha$. Since for critical refraction $\sin\alpha > 1/c$, additional asymptotics given in Ref. [11] must be used to determine the asymptotic behavior of $\tilde{\bar{u}}_{2p}^{*}$ and $\tilde{\bar{v}}_{2p}^{*}$, the double Fourier transforms of the radial and angular components of the reflected wave form dilatational velocities. When $\alpha > \sin^{-1}(1/c)$, these asymptotics and the method of stationary phase imply

$$\begin{bmatrix} \tilde{\bar{u}}_{2p}^{*}(r,\theta,\omega) \\ \tilde{\bar{v}}_{2p}^{*}(r,\theta,\omega) \end{bmatrix} \sim \frac{\sigma_0 H_{2p}(s_0) e^{i\omega t_{2p}}}{-i\omega} \sqrt{\frac{\cos\alpha}{2d_{2p}+\cos\alpha}} \begin{bmatrix} \cos\delta \\ \sin\delta \end{bmatrix} \quad \text{as} \quad \omega \to \infty, \tag{39}$$

where

$$\begin{aligned} H_{2p} &= H_{2p}^{R} + iH_{2p}^{I} \\ H_{2p}^{R} &= (C_{OR}C_{1R} + C_{OI}C_{1I})/(C_{1R}^2 + C_{1I}^2) \\ H_{2p}^{I} &= (C_{OI}C_{1R} - C_{OR}C_{1I})/(C_{1R}^2 + C_{1I}^2) \end{aligned}$$

where for $s\varepsilon[1/c, k_{1/c}]$, only the interior P wave is critically refracted, and

$$\begin{aligned} C_{OR}(s) &= s^2(e_2^2 - 4s^2 m_1 m_{k_2}) - 2\mu e_1 s^2(e_2^2 - 2m_1 m_{k_2}) - \mu\left(\frac{k_1 k_2}{c^2}\right)^2 m_1 m_{k_{1/c}} \\ &\quad -\mu^2 e_1^2(m_1 m_{k_2} - s^2), \\ C_{OI}(s) &= m_{1/c}[m_{k_{1/c}}(e_2^2 - 4s^2 m_1 m_{k_2}) - 4\mu m_{k_{1/c}} s^2(e_2 - 2m_1 m_{k_2}) + \mu m_{k_2}\frac{k_1^2 k_2^2}{c^2} \\ &\quad -4s^2\mu^2 m_{k_{1/c}}(m_1 m_{k_2} - s^2)], \\ C_{1R}(s) &= s^2(e_2^2 + 4s^2 m_1 m_{k_2}) - 2\mu e_1 s^2(e_2 + 2m_{k_2} m_1) + \mu\frac{k_1^2 k_2^2}{c^2} m_1 m_{k_{2/c}} \\ &\quad + \mu^2 e_1^2(m_1 m_{k_2} + s^2), \\ C_{1I}(s) &= m_{1/c}[m_{k_{1/c}}(e_2^2 + 4s^2 m_1 m_{k_2}) - 4\mu s^2 m_{k_{1/c}}(e_2 + 2m_1 m_{k_2}) \\ &\quad + \mu m_{k_2}\left(\frac{k_1 k_2}{c}\right)^2 + \mu^2 4s^2 m_{k_{1/c}}(m_1 m_{k_2} + s^2)]. \end{aligned}$$

However, for $s\varepsilon[k_1/c, 1)$, both the interior P and SV waves are critically refracted, and

$$\begin{aligned} C_{OR}(s) &= (s^2 - m_{1/c} m_{k_{1/c}})(e_2^2 - 4s^2 m_1 m_{k_2}) + 2s^2\mu(e_1 - 2m_{1/c} m_{k_{1/c}})(e_2 - 2m_1 m_{k_2}) \\ &\quad + \mu^2(4s^2 m_{1/c} m_{k_{1/c}} - e_1^2)(m_1 m_{k_2} - 2s^2), \\ C_{OI}(s) &= \mu\left(\frac{k_1 k_2}{c}\right)^2 (m_{k_2} m_{1/c} - m_1 m_{k_{2/c}}), \\ C_{1R}(s) &= (s^2 - m_{1/c} m_{k_{1/c}})(e_2^2 + 4s^2 m_1 m_{k_2}) - 2s^2\mu(e_1 - 2m_{1/c} m_{k_{1/c}})(e_2 + 2m_1 m_{k_2}) \\ &\quad + \mu^2(e_1^2 - 4s^2 m_{1/c} m_{k_{1/c}})(m_1 m_{k_2} + s^2), \\ C_{1I}(s) &= \mu\left(\frac{k_1 k_2}{c}\right)^2 (m_{k_2} m_{1/c} + m_1 m_{k_{1/c}}). \end{aligned}$$

m_x, e_1 and e_2 are given in (19). In addition, s_0 is such that

$$-2\alpha + \delta + \theta = 0 \tag{40}$$

where

$$\alpha = \sin^{-1} s_0, \qquad \delta = \sin^{-1}(s_0/r), \qquad s_0 > \sin(1/c).$$

Equation (40) has a simple ray interpretation where α is the angle of incidence and δ is the angle between the reflected ray to an exterior point (r, θ) and the radius vector r. d_{2p} is equal to $r \cos \delta - \cos \alpha$ and t_{2p} equals $d_{2p} + d_1 - 1$.

The Watson-type lemma and (39) imply

$$\begin{bmatrix} \dot{u}^*_{2p}(r, \theta, t) \\ \dot{v}^*_{2p}(r, \theta, t) \end{bmatrix} \sim \sigma_0 \sqrt{\frac{\cos \alpha}{2d_{2p} + \cos \alpha}} \begin{bmatrix} \cos \delta \\ \sin \delta \end{bmatrix} \left\{ H^R_{2p}(s_0) H(t - t_{2p}) + H^I_{2p}(s_0) \frac{\ln}{\pi} |t - t_{2p}| \right\}, \qquad \text{as } t \to t_{2p}, \tag{41}$$

for $\alpha > \sin^{-1}(1/c)$. For α and θ negative the evenness and oddness of $\dot{u}^*_{2p}$ and $\dot{v}^*_{2p}$ is invoked. The asymptotic behavior of the reflected shear velocities $\dot{u}^*_{2s}$ and $\dot{v}^*_{2s}$ are similarly calculated and are

$$\begin{bmatrix} \dot{u}^*_{2s}(r, \theta, t) \\ \dot{v}^*_{2s}(r, \theta, t) \end{bmatrix} \sim \sqrt{\frac{\sigma_0 k_2 \cos \alpha \cos \kappa}{\frac{d_{2s}}{k_2}(\cos \alpha + k_2 \cos \kappa) + \cos^2 \kappa}} \begin{bmatrix} \sin \varepsilon \\ \cos \varepsilon \end{bmatrix} \left\{ H^R_{2s}(s_0) H(t - t_{2s}) + H^I_{2s}(s_0) \frac{\ln}{\pi} |t - t_{2s}| \right\} \qquad \text{as } t \to t_{2s}, \tag{42}$$

for $\alpha > \sin^{-1}(1/c)$. Where s_0 is such that

$$\varepsilon + \theta - \alpha - \kappa = 0,$$

$$\alpha = \sin^{-1} s_0, \qquad \kappa = \sin^{-1}(s_0/k_2) \quad \text{and} \quad \varepsilon = \sin^{-1}(s_0/rk_2).$$

Also,

$$H^R_{2s}(s) = (g_{OR} C_{1R} + g_{IR} C_{1I})/(C^2_{1R} + C^2_{1I})$$

and

$$H^I_{2s}(s) = (g_{OR} C_{1R} - g_{OR} C_{1I})/(C^2_{1R} + C^2_{1I}).$$

If $s_0 \, \varepsilon [1/c, k_1/c]$ then

$$g_{OR}(s) = -2s(s^2 e_2 - \mu e_1(2s^2 + e_2) + \mu^2 e_1^2),$$
$$g_{OI}(s) = -4s m_{1/c} m_{k_1/c}[e_2 - 2s^2(1 + \mu)](1 - \mu).$$

If $s_0 \, \varepsilon [k_{1/c}, 1)$, then

$$g_{OR}(s) = 4s e_2(m_{1/c} m_{k_1/c} - s^2) - 4s\mu(2m_{1/c} m_{k_1/c} - e_1)(2s^2 + e_2) + 2\mu^2 s(4s^2 m_{1/c} m_{k_1/c} - e_1^2),$$
$$g_{OI}(s) = 0.$$

Physically this is analogous to the reflected P ray, where κ is the angle of reflection of the SV ray and ε is the angle between r, the vector to the point of interest, and the reflected SV ray. Also,

$$d_{2s} = r\cos\varepsilon - \cos\kappa,$$
$$t_{2s} = k_2 d_{2s} + d_1 - 1.$$

The above applies for $\alpha > \sin^{-1}(1/c)$. It is of interest to note that when transitional asymptotics for the Bessel functions are used for the cases $s_0 = 1/c$ or $s_0 = k_{1/c}$ the same results are found as those given above. Hence, (41) actually holds for $\sin^{-1}(1/c) \le \alpha \le \sin^{-1}(k_1/c)$ and (42) holds for $\sin^{-1}(k_1/c) \le \alpha < \pi/2$.

Lastly, as representative of singular waves that are refracted into the exterior by internally refracted waves, the contributions of $\dot{u}^*_{2ppp}$, $\dot{v}^*_{2ppp}$, $\dot{u}^*_{2pps}$ and $\dot{v}^*_{2pps}$, the dilatation and shear waves generated when the refracted P ray strikes the interface for the second time, are given. Recall that the interior P wave focuses after striking the interface for the first time and has a logarithmic singularity at its wave front the second time it strikes the interface. Thus, it is found

$$\begin{bmatrix}\dot{u}^*_{2ppp}(r,\theta,t)\\ \dot{v}^*_{2ppp}(r,\theta,t)\end{bmatrix} \sim \frac{\sigma_0}{\pi} H_{2ppp}(s_0)\begin{bmatrix}\cos\delta\\ \sin\delta\end{bmatrix}\ln|t - t_{2ppp}| \qquad \text{as } t \to t_{2ppp}, \tag{43}$$

and

$$\begin{bmatrix}\dot{u}^*_{2pps}(r,\theta,t)\\ \dot{v}^*_{2pps}(r,\theta,t)\end{bmatrix} \sim \frac{\sigma_0}{\pi} H_{2pps}(s_0)\begin{bmatrix}-\sin\varepsilon\\ \cos\varepsilon\end{bmatrix}\ln|t - t_{2pps}| \qquad \text{as } t \to t_{2pps}. \tag{44}$$

In (43), s_0 is such that

$$-2\alpha + \delta + \theta + 4\beta = 2\pi$$

and in (44), s_0 is such that

$$-\alpha - \kappa + \varepsilon + \theta + 4\beta = 2\pi$$

where α, β, δ, ε and κ are given in (40) and following (42). In addition,

$$H_{2ppp|s=s_0} = \left[\frac{\cos\beta}{2d_{2ppp}\left(2c - \dfrac{\cos\beta}{\cos\alpha}\right) + 4c\cos\alpha - \cos\beta}\right]^{1/2} \cdot \frac{\delta_{11}}{\delta_{10}}(c_{31} - c_{11}\delta_{11})|_{s=s_0}$$

and

$$H_{2pps|s=s_0} = \frac{k_2\cos\alpha\cos\kappa}{\left[\cos\kappa(d_{2pps} + \cos\kappa)\left(4c\dfrac{\cos\alpha}{\cos\beta} - 1\right) - d_{2pps}\dfrac{\cos\alpha}{k_2}\right]^{1/2}} \cdot \frac{\delta_{11}}{\delta_{10}}(g_{01} - g_{02})|_{s=s_0}$$

where

$$c_{11}(s) = (m_{1/c}m_{k_1/c} + s^2)(e_2^2 - 4s^2 m_1 m_{k_2}) - 2s^2\mu(2m_{1/c}m_{k_1/c} + e_1)(e_2 - 2m_1 m_{k_2})$$
$$+ \frac{\mu k_1^2 k_2^2}{c^2}(m_{1/c}m_{k_2} - m_{k_1/c}m_1) - \mu^2(e_1^2 + 4s^2 m_1 m_{k_2})(m_1 m_{k_2} - s^2),$$

$$c_{31}(s) = (s^2 - m_{1/c}m_{k_1/c})(e_2^2 - 4s^2 m_1 m_{k_2}) + 2s^2\mu(2m_{1/c}m_{k_1/c} - e_1)(e_2 - 2m_1 m_{k_2})$$
$$-\frac{\mu k_1^2 k_2^2}{c^2}(m_{1/c}m_{k_2} + m_{k_1/c}m_1) - \mu^2(e_1^2 - 4s^2 m_1 m_{k_2})(m_1 m_{k_2} - s^2),$$

$$g_{01}(s) = -(m_{1/c}m_{k_1/c} + s^2)4se_2 + \mu s(4s^2 + 2e_1)(2m_{1/c}m_{k_1/c} + e_1)$$
$$-\mu^2 2s(e_1^2 + 4s^2 m_{1/c}m_{k_1/c}),$$

$$g_{02}(s) = (m_{1/c}m_{k_1/c} - s^2)4se_2 - \mu s(4s^2 + 2e_1)(2m_{1/c}m_{k_1/c} - e_1)$$
$$-2s\mu^2(e_1^2 - 4s^2 m_{1/c}m_{k_1/c}),$$

and m_x, e_α, δ_{11}, δ_{10} are given in equation (19). Also,

$$d_{2ppp} = r\cos\delta - \cos\alpha$$
$$d_{2pps} = r\cos\varepsilon - \cos\kappa$$
$$t_{2ppp} = d_{2ppp} + \frac{4\cos\beta}{c} + d_1 - 1$$
$$t_{2pps} = \frac{d_{2pps}}{k_2} + \frac{4\cos\beta}{c} + d_1 - 1,$$

where these distances and times have the usual physical meaning.

Additional results

The following results are discussed in detail in Ref. [11].

(1) By considering the radial component of velocity in the interior after its wave front has reflected n times from the interface, it is found that for n large the point where the ray touches the caustic asymptotically approaches the midpoint of the ray and hence focusing takes place as far as possible from the interface. Also, for n large, the wave front alternately focuses after an odd number of reflections and unfocuses (i.e. has a simple step discontinuity at its wave front) after an even number of reflections as the wave propagates in θ. And lastly, the magnitude of the coefficient multiplying the step discontinuity/logarithmic singularity decreases as $|d|^n$ where $|d| < 1$ because of energy lost into the exterior at each reflection. Hence, it is clear that the effects of later arriving waves which result from a large number of reflections are not as important as the effects of those that have reflected only a few times.

(2) By use of the convolution theorem and the solution of the previous sections it can be shown that if the incident stress pulse had been continuous instead of a step function then all of the logarithmic singularities would instead be bounded functions. Thus a necessary condition for the existence of a propagating infinite discontinuity in stress is that the incident stress pulse have at least a step discontinuity at its wave front. A physical stress pulse is always continuous; thus, the results found here are to be interpreted as a limiting case as the rise time of a continuous incident pulse becomes small. In fact, it is for this reason that the coefficients which multiply the logarithmic singularities are important. As the rise time of the incident stress pulse becomes small, the focused response in the neighborhood of the wave fronts, while still finite, are proportional to these coefficients and, hence, they determine relative stress levels.

(3) The contributions to the velocity of a point on the interface from the diffracted dilatation and shear waves and the Stonely (Rayleigh-like) interface wave were evaluated[11]

using an alternative representation of the solution found by exchanging the Fourier integrals of (12) for a residue type evaluation. The results were evaluated asymptotically for large ω and were found to be of small order ω^{-1} and hence continuous at their wavefronts from the Watson-type lemma. By continuity at the interface, any waves refracted into the interior also have a continuous wave front and hence, as mentioned above, cannot after focusing propagate an infinite stress discontinuity at their wave fronts. Thus the effects of these waves compared to the focused, refracted waves are negligible.

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Абстракт — Обсуждается задача плоской деформации для импульса напряжений, ударяющего упругое круглое цилиндрическое включение, погруженное в бесконечной упругой сплошной среде. Принятый метод определяет сингулярности преобладающих напряжений, которые возникают на фронтах волн от фокусировки волн преломленных во внутренней области. Находится, что необходимым и достаточным условием для существования сингулярности распространяющегося напряжения является факт ступеньчатого разрыва падающего импульса на его фронте. Определяется подробно поведение фронта волн для несколько первых волн давления и сдвига по направлении фокуса. Указано также, что добавление от других волн является менее важным. Во внешней области находится, что в большинстве составленных материалов отраженные волны имеет сингулярность на своих фронтах волн, которая зависит от угла отражения. Дается, также, подробно поведение несколько первых сингулярных пропулценных волн.

Анализ основан на приложении леммы типа Ватсона, развинутой здесь, и на методе Фридландера. Лемма относит асимптотическое поведение решения на фронте волны к асимптотическому поведению его преобразования Фурье по времени, для больших значений параметра преобразования. Применяется метод Фридландера, с целью представления решения в терминологии форм углово распространяющихся волн. Метод применяет интегральные преобразования, как по времени так и по коорданате по окружности θ. Определяется асимптотический интеграл инверсии θ для больших значений параметра преобразования по времени, путем подходящих асимптотик для функций Бессеаля и Ганкеля и метода стационарной фазы. Далее применяется лемма типа Ватсона, с целью определения решения на фронтах сингулярных волн.

Вообще, лемма типа Ватсона пригодна к задачам, которые вызывают сингулярные нагрузки и фокусировки, в которых является важным поведение фронта волны. Это дает поведение фронтов сингулярных волн, но так или иначе, сингулярная волна оказывается первой, которая появляется. Это применение обобщает метод Фридландера на внутренный район и физически объясняет, полученное представление с точки зрения теории излучения.